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A Comparative Study of Some Spatial-Temporal Discretization Schemes for Nonlinear Magnetohydrodynamic Simulation of Plasmas

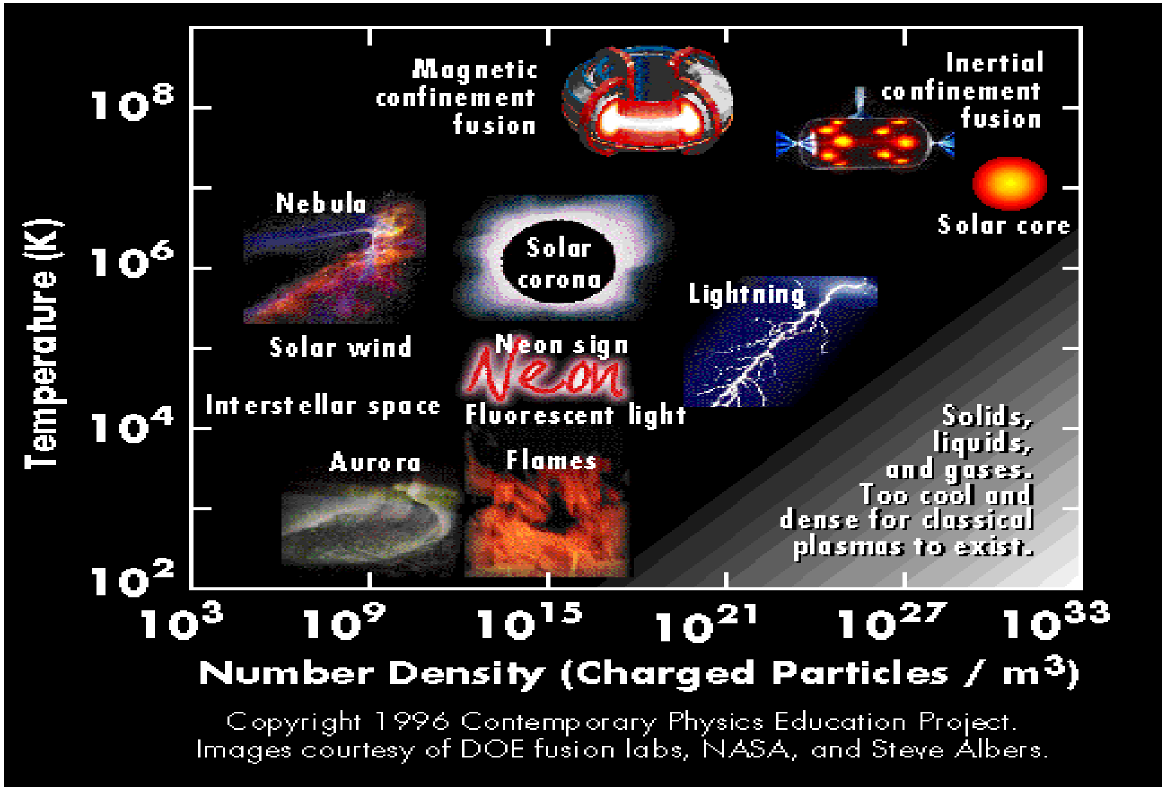
D. Nath, M. S. Kalra and P. Munshi

Nuclear Engineering and Technology Program
Indian Institute of Technology Kanpur
Kanpur - 208 016, India
(dnath@iitk.ac.in)

Outline

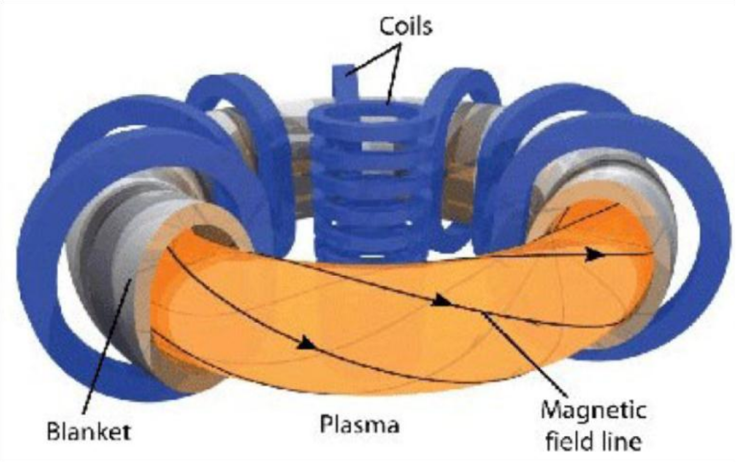
- Introduction
- MHD Model of plasma
    - Ideal MHD
    - One-dimensional Ideal MHD equations
    - Initial and Boundary Conditions
- Relevant Numerical Schemes and their Stabilities
- Numerical Results and Discussion
- Conclusion

Plasmas



- Plasmas are ubiquitous in nature having different densities and temperatures
- Fusion plasmas are required to have high temperature ($>10^8$ K ~ 10 keV)

Fusion Plasma



- Confining plasma using magnetic field in a torus i.e. the Tokamak, is the main candidate for achieving fusion on earth
- Due to collisionless nature, kinetic model (Boltzmann Equation) represent the correct behavior of the plasma
- However fluid model are as applicable in a limited time scales (~ ms) and length scales

Ideal Magnetohydrodynamic Equations

- Continuity equation: $$\frac{d\rho}{dt} + \rho\nabla\cdot\boldsymbol{v} = 0$$
- Momentum equation: $$\rho\frac{d\boldsymbol{v}}{dt} + \nabla\left(P + \frac{B^2}{8\pi}\right) = \frac{\boldsymbol{B}\cdot\nabla\boldsymbol{B}}{8\pi}$$
- Energy equation: $$\frac{dP}{dt} + \gamma\nabla\cdot\boldsymbol{v} = 0$$

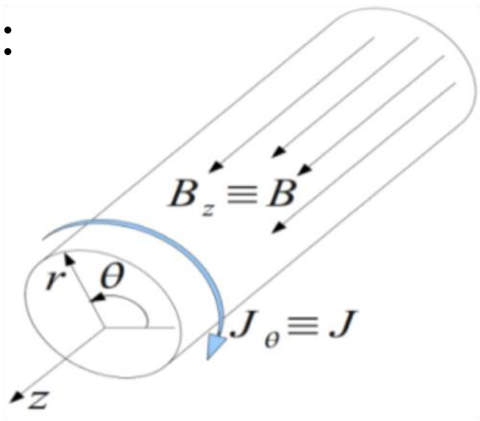
material derivative: $\dfrac{d}{dt} = \boldsymbol{v}\cdot\nabla + \dfrac{\partial}{\partial t}$

- Induction equation:

$$\frac{d\boldsymbol{B}}{dt} - \boldsymbol{B}\cdot\nabla\boldsymbol{v} + \boldsymbol{B}\nabla\cdot\boldsymbol{v} = \begin{cases} \dfrac{c^2\eta}{4\pi}\nabla^2\mathbf{B} & : \quad \textbf{RMHD} \\ 0 & : \quad \textbf{IMHD} \end{cases}$$

One-dimensional MHD Eqns

- MHD forms a hyperbolic system of 8 equations:
  - 2 scalar, 3+3 vector equations;
  - 8 variables: $\rho, v_r, v_\theta, v_z, P, B_r, B_\theta, B_z$



- In cylindrical coordinates 1-D reduction:
  - B is along z-direction
  - $B_r = 0$, $B_\theta = 0$ and $B_z = B$
  - assuming no angular rotation in the plasma ($v_\theta = 0$)
  - no motion along z- direction ($v_z = 0$) so remaining velocity is only along r- direction ($v_r = v$)
  - due to symmetry in θ and z – directions:

$$\frac{\partial}{\partial\theta} = 0 \text{ and } \frac{\partial}{\partial z} = 0$$

- Problem is reduced into one dimensional in cylindrical coordinates

One-dimensional MHD Equations

- One-dimensional MHD equations in Cylindrical coordinates
- Non-dimensionalzation to reach in Alfvén time scale

$$r \rightarrow ra, \quad t \rightarrow ta/v_A, \quad \rho \rightarrow \rho\rho_o, \quad v \rightarrow vv_A, \quad P \rightarrow PP_o, \quad B \rightarrow BB_a$$

Alfvén Speed $v_A = \sqrt{(B_a^2/\mu_0\rho_o)}$, $\beta = P_o/(B_0^2/2\mu_0)$

$$\frac{\partial\rho}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(r\rho v)$$
$$\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial r} - \frac{1}{\rho}\frac{\partial}{\partial r}\left(P + \frac{B^2}{2\mu_0}\right)$$
$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial r} - \gamma\frac{P}{r}\frac{\partial}{\partial r}(rv)$$
$$\frac{\partial B}{\partial t} = -\frac{B}{r}\frac{\partial}{\partial r}(rv) - v\frac{\partial B}{\partial r}$$

$$\frac{\partial\rho}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(r\rho v)$$
$$\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial r} - \frac{\beta}{2}\frac{1}{\rho}\frac{\partial}{\partial r}\left(P + \frac{B^2}{\beta}\right)$$
$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial r} - \gamma\frac{P}{r}\frac{\partial}{\partial r}(rv)$$
$$\frac{\partial B}{\partial t} = -\frac{B}{r}\frac{\partial}{\partial r}(rv) - v\frac{\partial B}{\partial r}$$

Matrix Form of One-dimensional MHD Equations

- Ideal MHD set of equations can be written in matrix form

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial r} = \mathbf{S}(\mathbf{U}, r) \qquad \mathbf{U} = \begin{Bmatrix} \rho \\ v \\ P \\ B \end{Bmatrix} \qquad \mathbf{S}(\mathbf{U}, r) = \begin{Bmatrix} -\rho v/r \\ 0 \\ -\gamma v P/r \\ -vB/r \end{Bmatrix}$$

- Matrix **A(U)** and its eigen values:

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} v & \rho & 0 & 0 \\ 0 & v & \beta/2\rho & B/\rho \\ 0 & \gamma P & v & 0 \\ 0 & B & 0 & v \end{bmatrix} \qquad \Lambda \equiv |\lambda_A|_{max} = \sqrt{\frac{B^2}{\rho} + \frac{\beta\gamma P}{2\rho}} + |v|$$

- Positive roots implies the hyperbolic nature

Initial and Boundary Conditions

- Initial Conditions

$$\left.\begin{aligned} B(r,0) &= \sqrt{1-\beta(1-3r^2+2r^3)} \\ P(r,0) &= (1-3r^2+2r^3)[1+\delta\sin(2\pi r)] \end{aligned}\right\} \text{for } 0 \le r \le 1$$

$$Constant \quad \delta = 0.1$$

- Boundary Conditions

$$r=0, \qquad \frac{\partial \rho}{\partial r} = 0, \qquad v = 0, \qquad \frac{\partial P}{\partial r} = 0, \qquad \frac{\partial B}{\partial r} = 0$$

$$r=1, \qquad \rho = 0, \qquad v = 0, \qquad P = 0, \qquad B = 1$$

Numerical Schemes

- We have compared the stability of three numerical schemes in one dimension in cylindrical coordinates

  - One dimensional MHD equations
  - Cylindrical geometry
  - Finite Difference Method
  - Explicit time stepping

- The Lax-Wendroff-Retchmyer
- MacCormack
- Runge-kutta fourth order

Lax-Wendroff-Retchmyer Scheme

PDE:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = 0$$

Discretization:

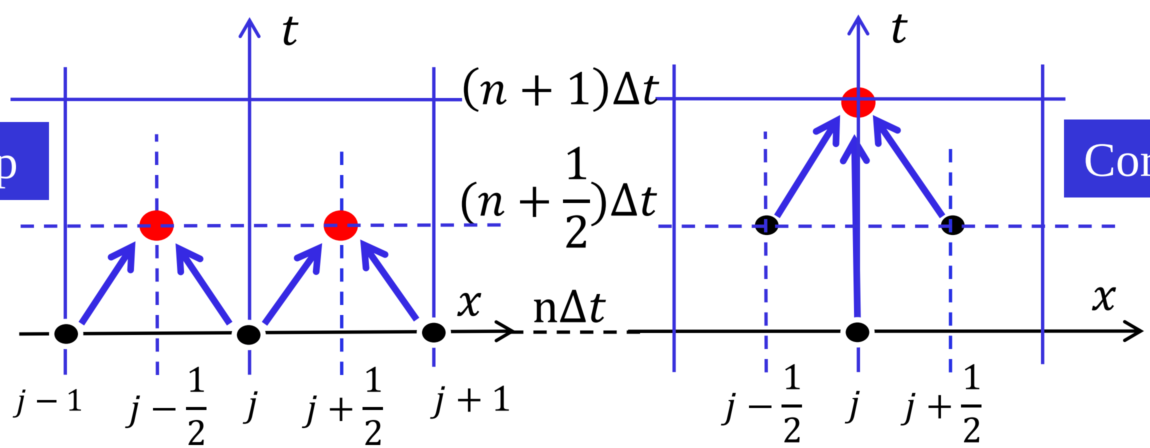
$$\boldsymbol{U}_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left[\boldsymbol{U}_{j+1}^{n} + \boldsymbol{U}_{j}^{n}\right] - \frac{\Delta t}{2\Delta x}\left[\boldsymbol{F}_{j+1}^{n} - \boldsymbol{F}_{j}^{n}\right]$$

: Predicator Step

$$\boldsymbol{U}_{j}^{n+1} = \boldsymbol{U}_{j}^{n} - \frac{\Delta t}{\Delta x}\left[\boldsymbol{F}_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \boldsymbol{F}_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right]$$

: Corrector Step

Predicator Step



Corrector Step

MacCormack Scheme

PDE:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = 0$$

Discretization:

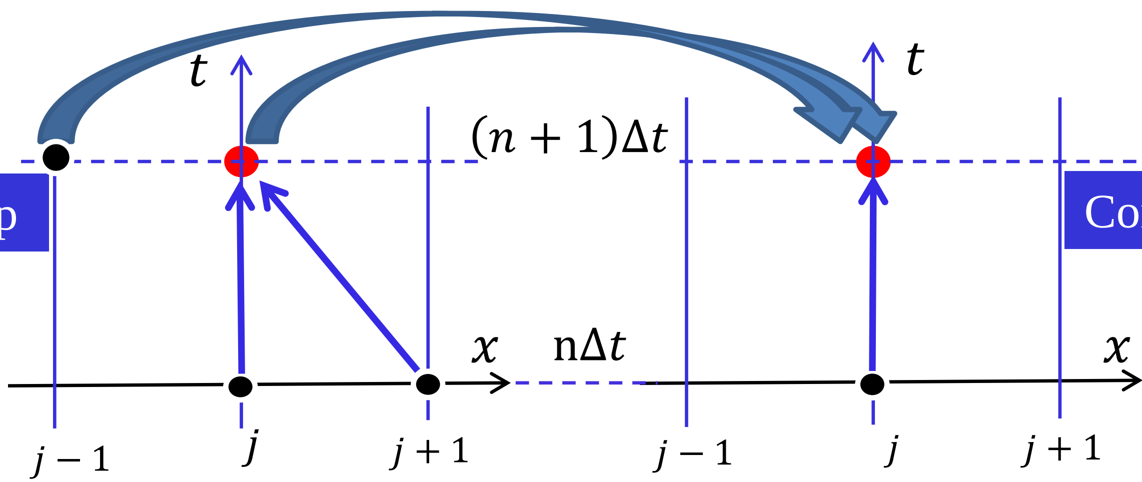
$$\overline{U_j^{n+1}} = \boldsymbol{U}_j^n - \frac{\Delta t}{\Delta x}\left[\boldsymbol{F}_{j+1}^n - \boldsymbol{F}_j^n\right]$$

: Predicator Step

$$\boldsymbol{U}_j^{n+1} = \frac{1}{2}\left\{\boldsymbol{U}_j^n + \overline{\boldsymbol{U}_j^{n+1}} - \frac{\Delta t}{\Delta x}\left[\overline{\boldsymbol{F}_j^{n+1}} - \overline{\boldsymbol{F}_{j-1}^{n+1}}\right]\right\}$$

: Corrector Step

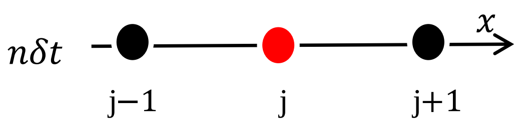
Predicator Step

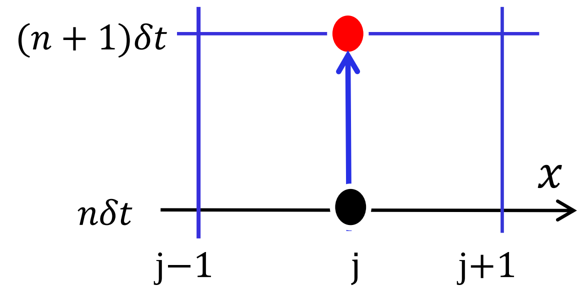


Corrector Step

Runge-Kutta Fourth Order

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = 0$$





- **Semi-Discritization for spatio-temporal PDE (Method of Lines):**
  - First discretized in space, obtain ODEs in time

$$\frac{dU_j(t)}{dt} = f(t, F_j)$$

  - Apply RK4 to advance in time

$$k^{n1} = f(t^n, F_j^n)$$
$$k^{n2} = f\left(t^n + \frac{1}{2}\Delta t, F_j^n + \frac{1}{2}\Delta t k^{n1}\right)$$
$$k^{n3} = f\left(t^n + \frac{1}{2}\Delta t, F_j^n + \frac{1}{2}\Delta t k^{n2}\right)$$
$$k^{n4} = \left(t^n + \Delta t, F_j^n + \Delta t k^{n3}\right)$$
$$U^{n+1} = U^n + \frac{\Delta t}{6}(k^{n1} + 2k^{n2} + 2k^{n3} + \mathrm{k}^{n4})$$

Stability Criteria

- **A scalar linear advection equation with advection speed Λ:** $\frac{\partial u}{\partial t} + \Lambda \frac{\partial u}{\partial r} = 0$

- **Von-Neumann stability analysis:** $u_k(x,t) = \hat{u}_k\, e^{i\phi}$ where $\phi = \omega \Delta t = k \frac{\Delta x}{\Delta t} \Delta t$

- **Numerical amplification Factor:** $|\mathrm{G}| = \frac{u_k^{n+1}}{u_k^n} \leq 1$ for stability

- **Lax-Wendroff-Retchmyer or MacCormak**

$$\mathrm{G} = \left(1 - \tfrac{1}{2}\mathrm{C}^2 \sin^2\phi + \tfrac{1}{24}\mathrm{C}^4 \sin^4\phi\right) + j\left(-\mathrm{C}\sin\phi + \tfrac{1}{6}\mathrm{C}^3 \sin^3\phi\right)$$

$$|\mathrm{G}| = \sqrt{1 - 4\mathrm{C}^2(1-\mathrm{C}^2)\sin^4(\phi/2)}$$

$$\boxed{\mathrm{C} = \frac{\Lambda.\Delta t}{\Delta r} \leq 1}$$

- **Runge-Kutta fourth order**

$$\mathrm{G} = (1 - \mathrm{C}^2 + \mathrm{C}^2\cos\phi) + j(-\mathrm{C}\sin\phi); \quad j = \sqrt{-1}$$

$$|\mathrm{G}| = \sqrt{1 - \tfrac{1}{72}\mathrm{C}^6\left(1 - \tfrac{1}{8}\mathrm{C}^2\sin^2\phi\right)\sin^6\phi}$$

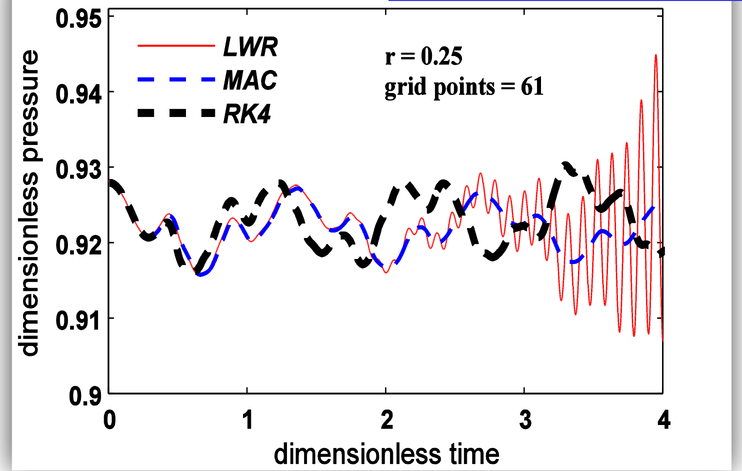
$$\boxed{\mathrm{C} = \frac{\Lambda.\Delta t}{\Delta r} \leq 2\sqrt{2}}$$

Numerical Results

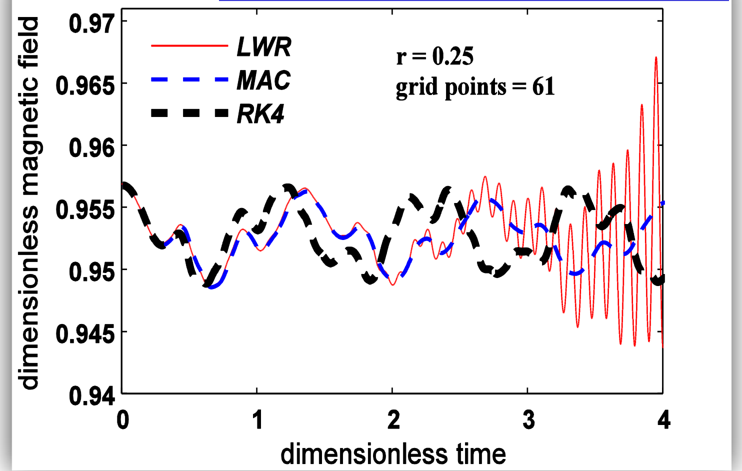
- Effect of Schemes
- Effect of Courant Number
- Effect of Grid Refinement
- Long Time Run
- Stability

Effect of Schemes

Pressure Vs time

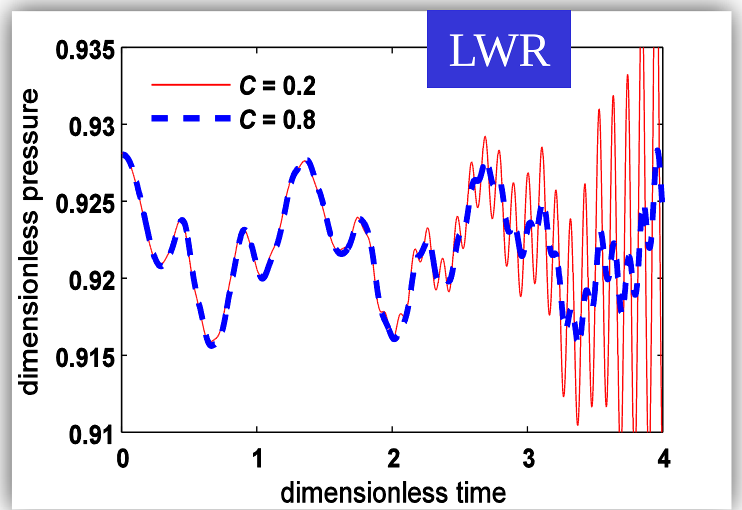


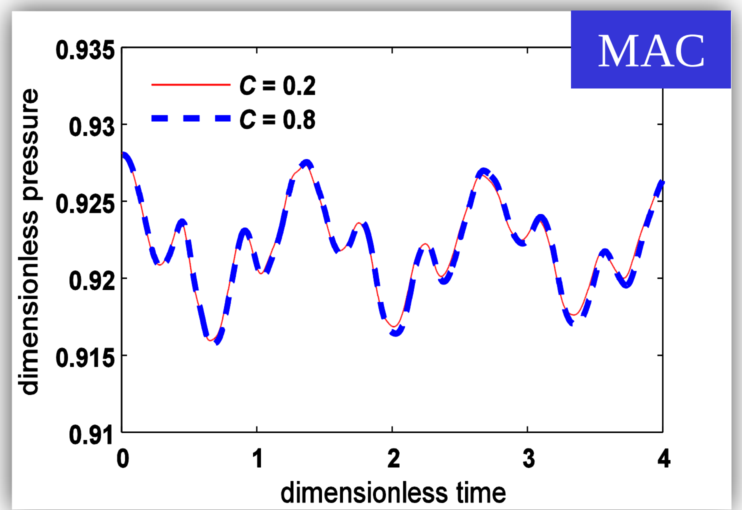
Magnetic Field Vs time

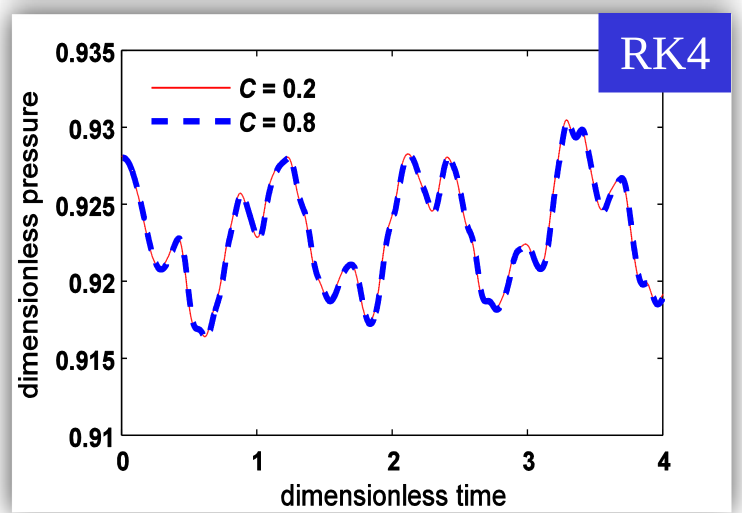


- LWR, MAC and RK4 schemes can advance more than 4 Alfven time scales for grid points 61 and $r = 0.25$
- LWR suffers from oscillations comparatively earlier and reaches high around time ~ 4 τ_a
- Similar behavior is observed in time stepping of Magnetic field

Effect of Courant Number

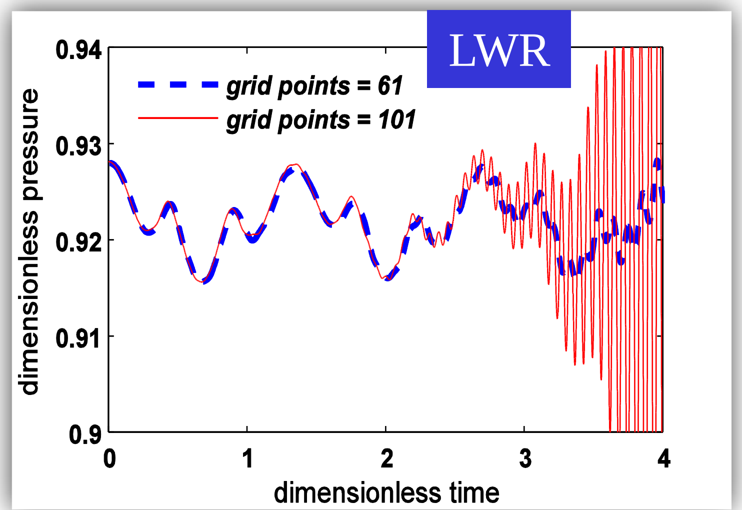


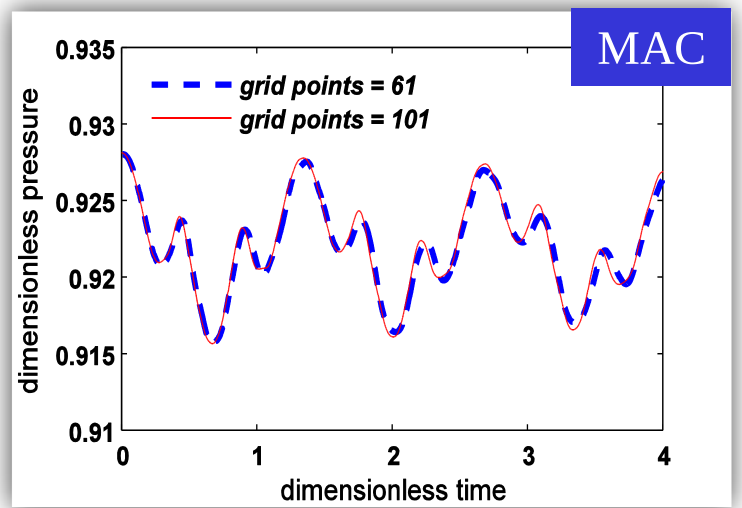


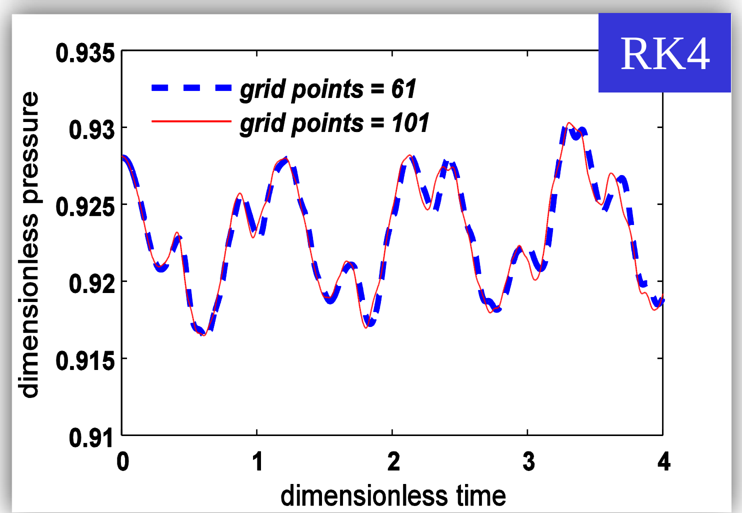


- Keeping fixed grid and changing Courant Number from C = 0.2 to C = 0.8 LWR suffers from oscillation from the early phases
- MAC or RK4 are smoother

Effect of Grid Refinement



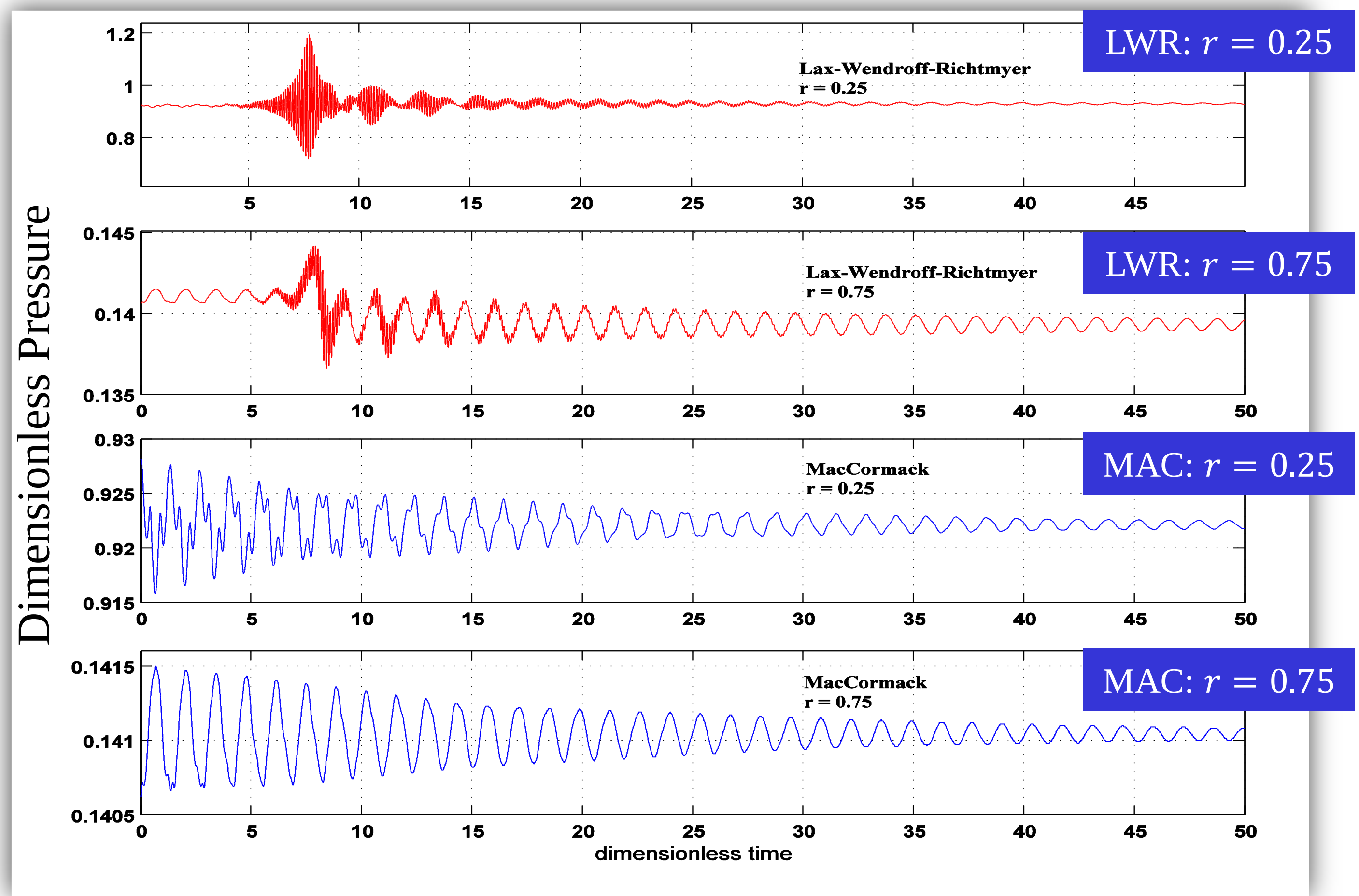




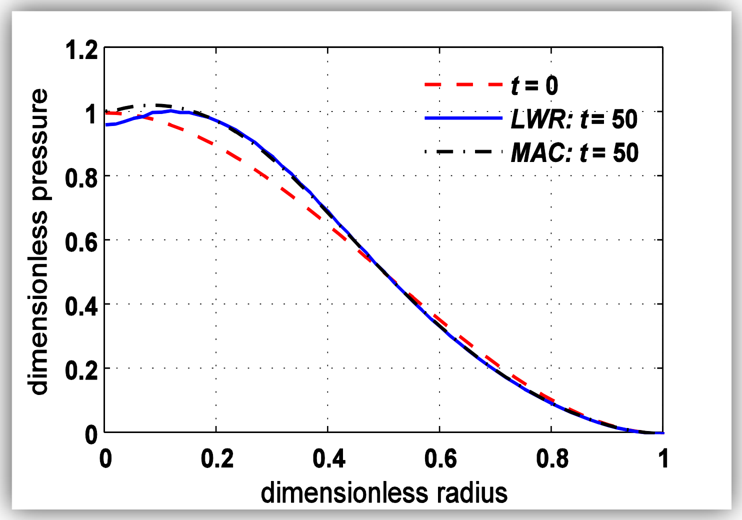
- On increasing the grid points again LWR surprisingly have lot of numerical instabilities whereas MAC and RK4 have reached grid independence.
- C = 0.8 (Kept Fixed)

Long Time Run

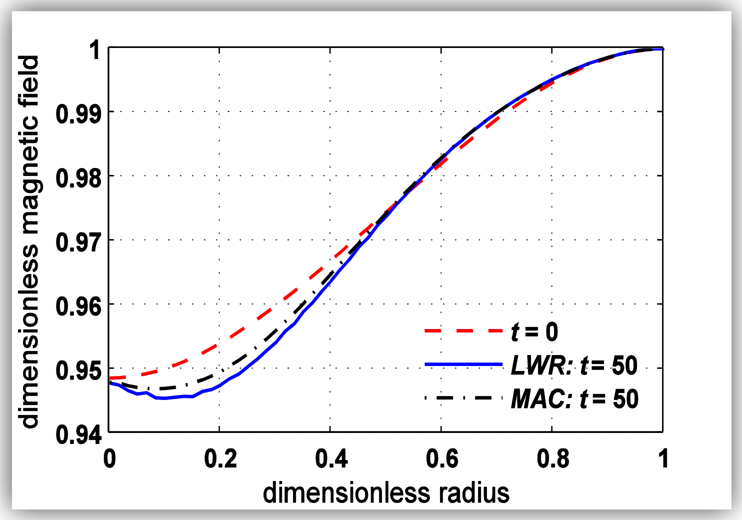
- From $t = 5\ to\ 15$, burst of instabilities appear in LWR $(C = 0.2,$ Grid points $= 61)$



Radial Distribution



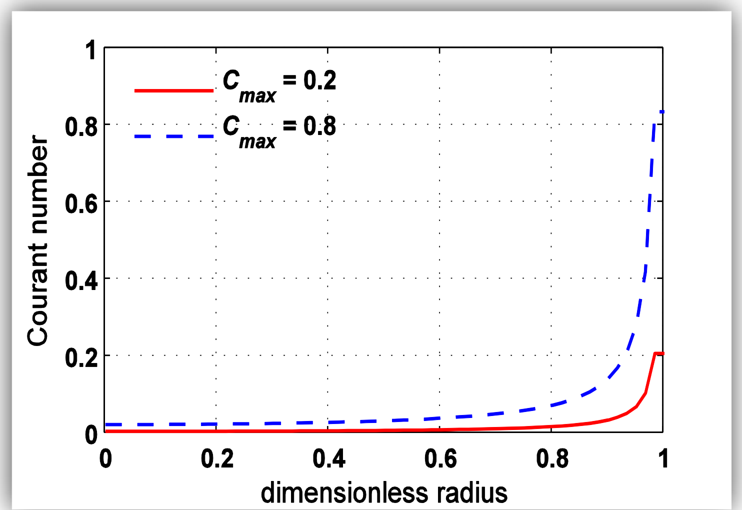
Pressure Vs Radius ($t = 50$)



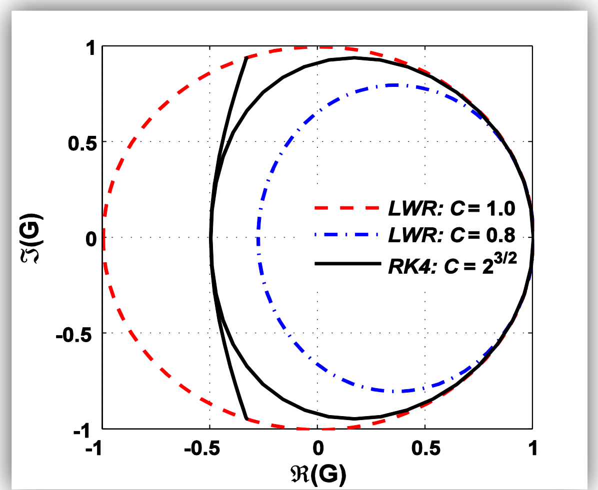
Magnetic Field Vs Radius ($t = 50$)

- A quasi-steady state is reached after t=50

Stability and Courant Number



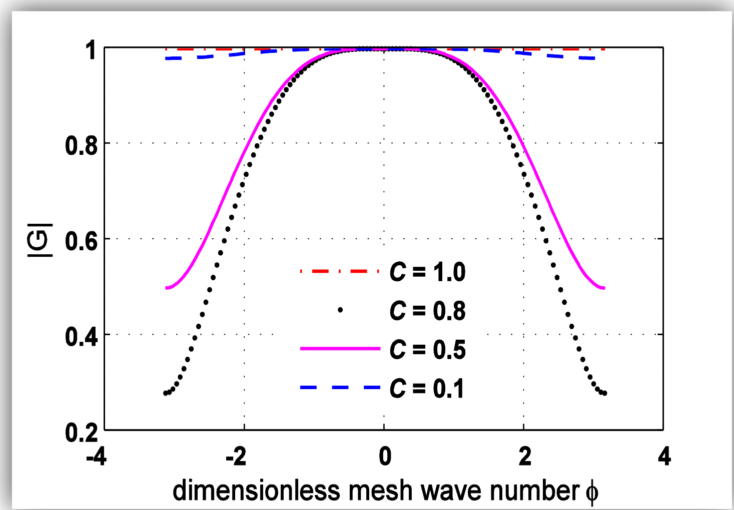
Radial Variation of Courant number for LWR



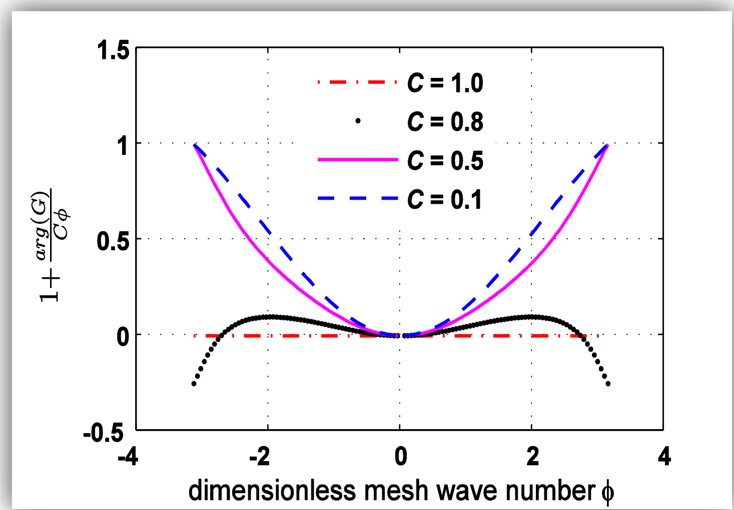
Stability Region of LWR and RK4

- The growth of numerical instabilities begins from the edge $r = 1$. At $r = 1$ Courant number become high, and if comes out from the stability region, makes an scheme numerically unstable
- Stability regions of LWR (with $C = 1.0, 0.8$) and RK4 ($C = 2\sqrt{2}$)
- Any scheme to be stable, needs to be operated within its stability region

Numerical Amplification and Phase Error in LWR



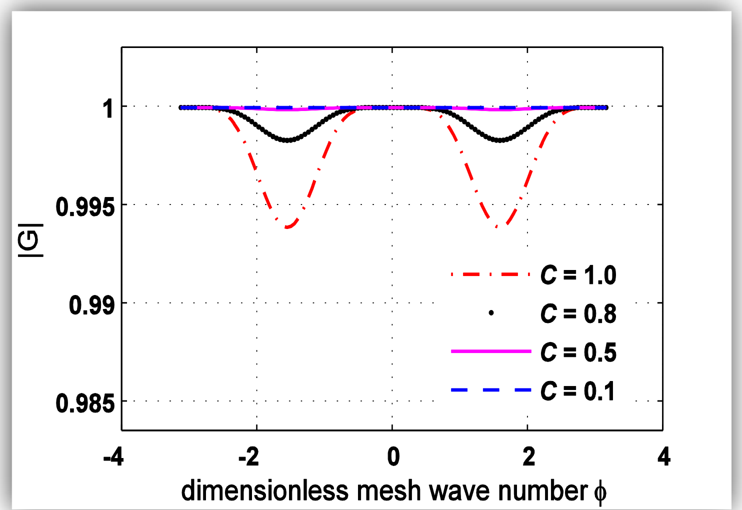
Numerical Dissipation LWR



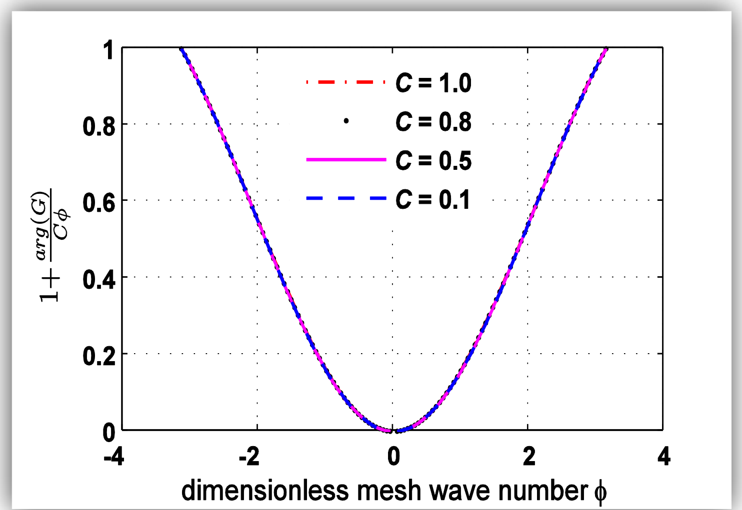
Phase Error for LWR

- Numerical amplification variation w.r.t the mesh wave number is negligible for Courant numbers near to $C = 1.0$ and $C = 0.1$ and maximum for $C = 0.8$
- Phase error for $C = 0.8$ become negative shows numerical instability in the scheme
- A positive dispersion (phase) error implies that numerical advection velocity is larger than the exact physical velocity and vice versa

Numerical Amplification and Phase Error in RK4



Numerical dissipation for RK4



Phase Error for RK4

- Less numerical dissipation in RK4 comparatively LWR and MAC
- Phase error is almost independent of the Courant number
- Phase error continuously increases towards the high wave numbers

Conclusion

- Three widely used discretization schemes applied to a representative *nonlinear* problem arising in the MHD simulation of plasmas: **LWR, MAC** and **RK4**
- **LWR** and **RK4** are not suitable schemes for nonlinear hyperbolic MHD equation
- **MAC** scheme always gave stable results if CFL is satisfied
- The linear stability conditions are **necessary** but **not sufficient** to guarantee the numerical stability of these algorithms when applied to a ***nonlinear*** **hyperbolic problem**
- The solution for the transient phase obtained from the three schemes are found to differ significantly due to different amount of **numerical diffusion and dispersion** present in each scheme

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Thank you!